The Topological Cage: A Search-Less Geometric Handshake for Medial Axis Injection

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***Abstract*—While data-driven methods dominate robotic perception, classical geometric approaches remain unparalleled in providing deterministic safety guarantees for navigation. A persistent challenge in topological planning is the "Injection Problem": safely connecting an arbitrary off-road coordinate to a skeletal highway (Medial Axis) without computationally expensive grid searches or unsafe Euclidean projections. We propose the Topological Cage, a search-less injection algorithm utilizing SDF-Derived Topological Skeletonization (SDTS). By evaluating the gradient magnitude along the Medial Axis, our method dynamically distinguishes internal corridors from external exits to establish a bounded topological enclosure around the obstacle space. Bi-directional gradient scouts are mathematically guaranteed to intersect this skeletal graph in finite time, bounding the search space to $O(1)$ operations upon intersection. This paper proves our intrinsic theoretical guarantees and demonstrates how geometric structure outperforms heuristic and learning-based planners in guaranteed max-min clearance navigation.**

***Index Terms*—Path Planning, Medial Axis, Signed Distance Fields, Computational Geometry, Autonomous Navigation**

I. Introduction

As learning-based paradigms increasingly dominate robotics, classical methods guaranteeing provable safety are often sidelined. In mission-critical navigation, skeletonization methods like the Medial Axis Transform (MAT) [5] maximize obstacle clearance. However, safely navigating onto this skeleton from an arbitrary open location—the "Injection Problem"—remains challenging. Traditional planners (BFS, A*, RRT) explore massive search spaces or yield sub-optimal, obstacle-grazing paths, while Artificial Potential Fields (APF) risk local minima [1]–[3].

We introduce the *Topological Cage* via SDF-Derived Topological Skeletonization (SDTS). Treating the environment as a continuous scalar field, we create a deterministic, bounded manifold that bypasses brute-force search entirely, utilizing local differential calculus to safely inject the robot into the global graph structure.

II. Methodology: SDF-Derived Topological Skeletonization

Our method constructs a navigational graph directly from the geometric properties of the Signed Distance Field (SDF) across four distinct phases.

A. Phase 1: Metric Formulation & Ridge Extraction

For any point x in free space Ω, we evaluate the SDF, $\Psi(x)$ [5], representing absolute Euclidean distance to the nearest obstacle boundary $\partial\mathcal{O}$ (Fig. 1). Rather than relying on morphological thinning, we extract the mathematical ridges of the distance field where the gradient of $\Psi(x)$ diverges. These local maxima inherently map to continuous Voronoi boundaries [6], geometrically guaranteeing a path at the global maximum of obstacle clearance (Fig. 2).

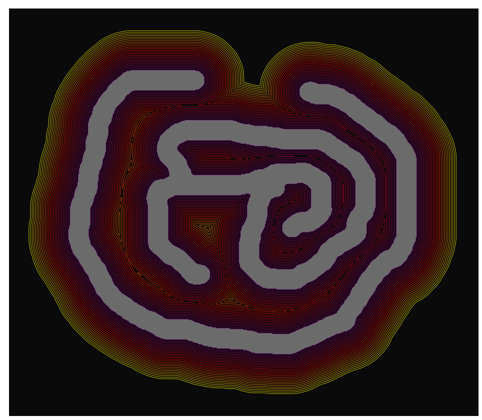

Fig. 1. Iso-contours of the SDF encoding Euclidean distance to obstacles.

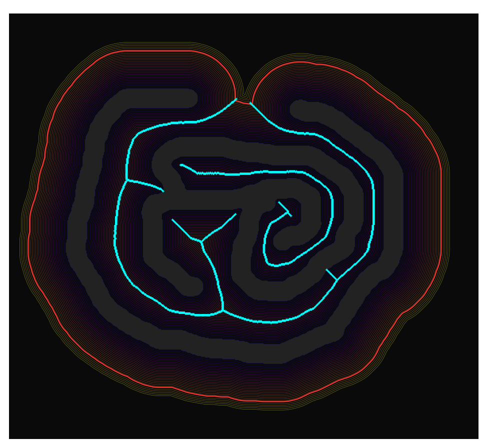

Fig. 2. The skeletal graph (cyan) derived directly from SDF local maxima.

B. Phase 2: The Topological Cage Construction

To prevent the Medial Axis from extending infinitely into open space, we dynamically project a bounding shell. Inside a corridor, SDF level sets run parallel to the Medial Axis, driving gradient magnitude near zero. We isolate these internal corridors M_{int} via threshold τ_{grad}:

$$M_{int} = \{p \in M \mid \|\nabla\Psi(p)\| \leq \tau_{grad}\} \quad (1)$$

We extract the maximum internal clearance C_{max} and define an outer closure limit $d_{cage} = C_{max} + \Delta_{offset}$, utilizing a heuristic fallback $\Psi_{default}$ for strictly open environments:

$$C_{max} = \begin{cases} \max_{p \in M_{int}} \Psi(p) & \text{if } M_{int} \neq \emptyset \\ \Psi_{default} & \text{otherwise} \end{cases} \quad (2)$$

The Topological Cage $\mathcal{C}$ is constructed by uniting pruned inner ridges ($\Psi(p) \leq d_{cage}$) with an outer contour shell of thickness ϵ. A morphological closing Φ_{close} [8] guarantees an unbroken loop:

$$\mathcal{C} = \Phi_{close}\Big(\{p \mid |\Psi(p) - d_{cage}| \leq \epsilon\} \cup \{p \in M \mid \Psi(p) \leq d_{cage}\}\Big) \quad (3)$$

C. Phase 3: Bi-Directional Scout Injection

Because $\mathcal{C}$ forms a closed loop, continuous gradient traversal must eventually intersect it. From the start and goal, we execute Euler integration along both the gradient and anti-gradient:

$$p_{k+1} = p_k \pm \alpha \frac{\nabla\Psi(p_k)}{\|\nabla\Psi(p_k)\|} \quad (4)$$

The injection terminates the moment either the "Climber" $(+)$ or "Diver" $(-)$ intersects $\mathcal{C}$ (Fig. 3).

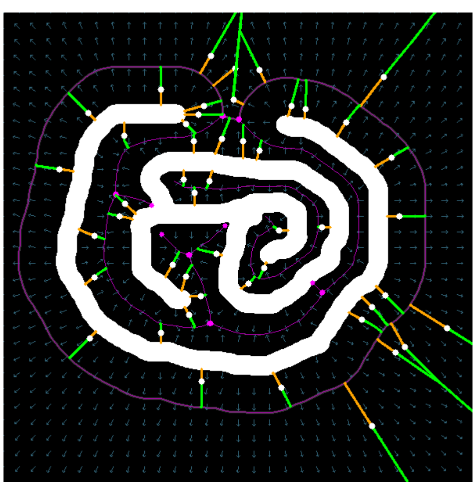

Fig. 3. Bi-Directional Scouts. Gradient vectors guide flow to intersect $\mathcal{C}$.

D. Phase 4: Graph Construction & Global Routing

Once scouts intersect $\mathcal{C}$, the continuous skeleton is abstracted into a discrete, undirected graph $G = (V, E)$ for global routing. The vertex set V comprises critical morphological points along the skeleton: junctions (> 2 neighbors), endpoints (1 neighbor), and scout intersection nodes ($p_{intersect}$).

The edge set E is formed by tracing continuous skeletal segments between critical nodes $v_i, v_j \in V$. The weight $w(v_i, v_j)$ is the cumulative Euclidean distance of the traced sequence of skeleton pixels $\{x_k\}_{k=1}^{N}$:

$$w(v_i, v_j) = \sum_{k=1}^{N-1} \|x_{k+1} - x_k\| \quad (5)$$

By converting the environment into this sparse topological graph, we drastically prune the search space. Evaluating only structural milestones instead of thousands of grid cells allows Dijkstra's algorithm [10] to compute shortest paths in negligible time.

III. Mathematical Guarantees

SDTS transitions from heuristic searches to calculus-based integration, providing deterministic geometric guarantees.

Theorem 1 (Topological Closure of the Cage): The Topological Cage $\mathcal{C}$ forms a closed boundary dividing bounded free space Ω into an interior region Ω_{int} and an exterior layer.

Proof: Because $\Psi(x)$ is Lipschitz continuous, level set $L = \{x \in \Omega \mid \Psi(x) = d_{cage}\}$ forms closed loops enclosing local maxima where $\Psi(x) > d_{cage}$. Since $\mathcal{C}$ includes this loop united with the skeletal graph, any continuous path ascending to maximum clearance from $\partial\mathcal{O}$ must intersect $\mathcal{C}$, guaranteeing closure. ■

Theorem 2 (Finite-Time Convergence): The gradient system $\dot{p} = \pm\nabla\Psi(p)$ intersects $\mathcal{C}$ in finite length, requiring $O(1)$ grid expansions upon intersection.

Proof: Since $\Psi(x)$ has no local minima in interior Ω aside from ridges, following the positive gradient strictly increases Ψ monotonically. In bounded Ω, flow converges onto the Medial Axis or intersects contour $\{\Psi(x) = d_{cage}\}$. By Theorem 1, both belong to $\mathcal{C}$. The streamline thus monotonically terminates at $\mathcal{C}$, independent of global map area. ■

IV. Experimental Results

We benchmarked SDTS against A* [9], PRM, RRT*, and the Fast Marching Method (FMM) in highly non-convex maps featuring U-shaped traps. As seen in Fig. 4, SDTS maximizes clearance by adhering to the medial axis, unlike A* and FMM (which graze obstacles), PRM (jagged paths), or RRT* (fails trap). SDTS consistently resolves arbitrary off-road queries by dynamically routing entry/exit points into $\mathcal{C}$ (Fig. 5).

A. Benchmark Performance

Global routing results are summarized in Table I and Fig. 6.

Computational Efficiency: By bypassing free-space exploration, SDTS evaluated only 98 nodes—orders of magnitude fewer than A* or FMM. Query time dropped to 28.38 ms ($> 30\times$ faster than A*). While SDTS requires a 189.25 ms one-time SDF preparation cost, its near-instantaneous query speed is optimal for static or multi-agent environments.

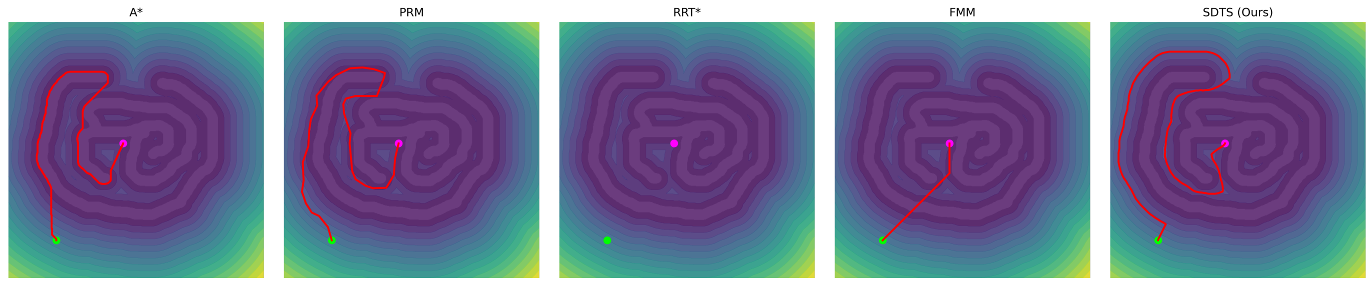

Fig. 4. SDTS (Ours) adheres to the medial axis to maximize clearance, while other methods graze obstacles, produce jagged trajectories, or fail.

Fig. 5. Sequential queries reliably routed into $\mathcal{C}$ via bi-directional scouts.

TABLE I
QUANTITATIVE BENCHMARK RESULTS

Algorithm	Nodes Visited	Clearance	Nav Time (ms)	Success
A*	244,021	1.000	3695.987	True
PRM	1,002	3.000	430.757	True
RRT*	1191	0.000	473.769	False
FMM	360,000	0.000	176.216	False
SDTS (Ours)	**98**	**14.0**	**28.38**	**True**

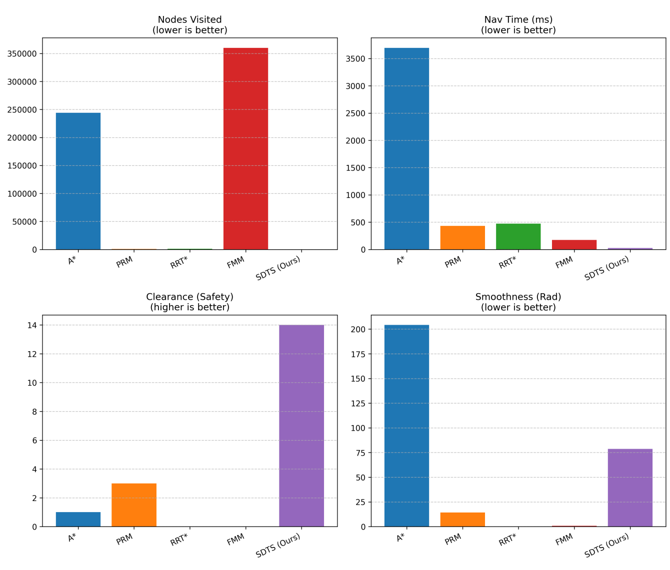

Fig. 6. SDTS reduces the evaluated search space while maximizing safety.

Max-Min Clearance (Safety): Gradient ascent guarantees immediate movement away from obstacles. SDTS achieved a clearance score of 14.0, outperforming PRM (3.0) and A* (1.0). Unlike RRT*, which often fails in narrow corridors, SDTS deterministically solves the maze.

V. CONCLUSION

We presented an adaptive, search-less injection framework bridging continuous local vector fields with discrete global graph searches. By utilizing fundamental theorems of manifolds and distance fields, SDTS dynamically generates a Topological Cage that eliminates heuristic grid searches. This demonstrates that structural geometry remains an indispensable, computationally efficient tool for intrinsically interpretable and provably safe mobile robotics.

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