Introducing Sylvester Forms to Robotics: Efficient Closed-Form Pose Estimation

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***Abstract*—Pose estimation from 3D-to-3D correspondences is a fundamental problem in robotics and computer vision, with strong relevance to real-time perception and localization. It is commonly formulated as a nonlinear optimization problem that can be reduced to a polynomial system and solved in closed form. In this paper, we introduce a new class of resultant-based polynomial solvers for pose estimation that exploits Sylvester forms to reduce the complexity of elimination. By integrating Sylvester forms into the hidden-variable formulation, we derive closed-form solvers operating in degrees 7 and 8, instead of degree 9, which leads to smaller elimination matrices and lower computational cost. Experiments on the KITTI dataset show that the proposed solvers are numerically accurate and faster than state-of-the-art closed-form approaches. Beyond the specific solver, our results highlight a broader point that is particularly relevant for geometric robotics: geometric methods and data-driven methods need not be opposed. While the solver itself is derived from exact algebraic structure, its numerical performance depends on implementation choices such as the order of monomials that induces the block decomposition of the elimination matrix. Since we currently do not have a principled method for selecting the ordering that gives the best numerical conditioning, this work suggests a promising hybrid direction in which offline learning is used to optimize these choices while preserving the exact geometric structure of the solver.**

I. INTRODUCTION

Pose estimation, i.e. the estimation of a rotation and a translation, is a fundamental problem in robotics, with classical solutions dating back to the early formulation of 3D registration [7]. Registering two sets of 3D points is usually expressed as a non-linear optimisation problem after matching points to points, points to planes or points to lines. A unified formulation of this optimisation problem has been introduced in [11].

Reducing the optimisation problem to a polynomial system solving one allows it to be solved in closed form, providing a more robust and predictable solution than direct iterative techniques [9], [10], [13], [14]. These approaches often rely on Gröbner basis methods or resultant constructions to eliminate unknowns and recover the camera pose.

In this work, we use a resultant-based solver following [9], [10]. Furthermore, we exploit Sylvester forms, initially introduced in [8, §3.10]. These have recently been revisited and generalised to the multigraded setting in [3], with potential applications in solving zero-dimensional polynomial systems.

Our contribution is to integrate Sylvester forms with the hidden-variable formulation of the resultant to develop new methods operating in degrees 7 and 8. This significantly reduces the size of the resultant matrices compared to the degree 9 approach in [10], thereby reducing computational costs while preserving algebraic completeness and numerical stability.

More broadly, our approach suggests that geometric and data-driven methods should be viewed as complementary rather than competing. The proposed solver is derived from exact algebraic structure, but its practical numerical performance depends on implementation choices such as the order of monomials that induces a block matrix decomposition. At present, we do not have a theoretical criterion for selecting the ordering that yields the best numerical conditioning. This makes the problem a natural candidate for offline learning, where representative data could be used to guide these choices while preserving the exactness and interpretability of the geometric solver. In this sense, the paper not only introduces Sylvester forms to robotics pose estimation, but also provides a concrete example of how geometric structure and learning-based heuristics may be combined in a principled way.

II. PROBLEM STATEMENT

A. Pose estimation as a polynomial system solving

The objective of the problem is to find the pose (rotation and translation) from measured points and corresponding points, lines and planes. The translation appears quadratically in the cost function and can be eliminated, resulting in a minimisation problem that depends only on the rotation [9]–[11], [13], [14]. Parametrising the rotation by a unit quaternion $\mathbf{q} = [w; x; y; z]$ and using Lagrange multipliers, we can impose the unit quaternion condition $\mathbf{q}^T \mathbf{q} = 1$. The minimisation problem is then reduced to a problem of finding real solutions of a polynomial system

$$e_1(\mathbf{q}, \lambda) = e_2(\mathbf{q}, \lambda) = e_3(\mathbf{q}, \lambda) = e_4(\mathbf{q}, \lambda) = 0 \qquad (1)$$

where the 4 polynomial equations depend linearly on λ and are homogeneous of degree 3 with respect to $\mathbf{q}$ [10]. Let

$$I = (e_1, e_2, e_3, e_4) \subset \mathbb{C}[\lambda][w, x, y, z] \qquad (2)$$

be the graded ideal generated by the four polynomials e_1, e_2, e_3, e_4, we denote by $V(I)$ the algebraic variety associated to I, i.e. the common roots of (1) [4]. For $d \in \mathbb{Z}$, we denote by $(I)_d$ the graded component of I of degree d.

The parameter λ can be eliminated from the equations (1). This yields six polynomial equations of homogeneous degree 4 in $\mathbf{q}$:

$$f_1(\mathbf{q}) = f_2(\mathbf{q}) = \cdots = f_6(\mathbf{q}) = 0 \qquad (3)$$

The problem of pose estimation can be seen as the problem of finding real points of the variety $\mathrm{V}(J) \subset \mathbb{P}^3_{\mathbb{C}}$, where

$$J = (f_1, \ldots, f_6) \subset \mathbb{C}[w, x, y, z] \qquad (4)$$

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There is a one to one correspondence between the points of $V(I)$ and $V(J)$. The following proposition (the proof can be found in [12]) gives an upper bound on the number of the solutions.

Proposition 1: If $V(J) \subset \mathbb{P}^3_{\mathbb{C}}$ is finite, then it consists of 40 points counted with multiplicity and with coordinates in $\mathbb{C}$.

B. Closed-form solution

A closed-form solution of the equations (1) based on the *hidden variable* approach (see [4, Chapter 3, §5] and [1]) is proposed in [10]. The polynomials (1) are seen as polynomials in $\mathbf{q}$, λ being interpreted as a parameter.

Notation 2: For a given degree $d \in \mathbb{N}$, we denote the vector of homogeneous monomials of degree $d \in \mathbb{N}$ in w, x, y, z by

$$\mathbf{m}_d = [w^{d_w} x^{d_x} y^{d_y} z^{d_z} : d_w + d_x + d_y + d_z = d] \quad (5)$$

The number of such monomials is $n_d = \binom{d+3}{d}$.

We briefly review the closed-form solution proposed in [10]. While the original work sets a certain degree d to 9, we present a generalized review for any value of d and state the necessary conditions it must satisfy.

First, $4 \cdot n_{d-3}$ equations are constructed:

$$\mathbf{e}(\mathbf{q}, \lambda, \mathbf{c}) \otimes \mathbf{m}_{d-3} = \mathbf{E}_d(\mathbf{c}, \lambda)\ \mathbf{m}_d^T = 0 \quad (6)$$

where $\mathbf{E}_d(\mathbf{c}, \lambda)$ is a $(4 \cdot n_{d-3}) \times n_d$ coefficient matrix, which depends linearly on λ and on the coefficients $\mathbf{c}$ of the equations (1).

Then, additional $6 \cdot n_{d-4}$ equations that do not depend on λ are considered:

$$\mathbf{f}(\mathbf{q}, \mathbf{c}) \otimes \mathbf{m}_{d-4} = \mathbf{F}_d(\mathbf{c})\ \mathbf{m}_d^T = 0 \quad (7)$$

where $\mathbf{F}_d(\mathbf{c})$ is a $(4 \cdot n_{d-4}) \times n_d$ coefficient matrix. In order to find the solutions, we need $\mathbf{F}_d(\mathbf{c})$ to have sufficient rank.

Proposition 3: If $d \geq 7$, $\text{rank}(\mathbf{F}_d(\mathbf{c})) = n_d - 40$.

Moreover, the following two conditions on $\mathbf{E}_d(\mathbf{c}, \mathbf{q})$ must be satisfied:

1) $\text{rank}(\mathbf{E}_d(\mathbf{c}, \lambda)) = n_d$ for general values of $\mathbf{c}$ and λ,
2) $\text{rank}(\mathbf{E}_d(\mathbf{c}, \lambda)) < n_d$ for some $\mathbf{c}$ and λ if and only if the corresponding polynomial system has solutions in $\mathbb{P}^3_{\mathbb{C}}$.

Then we can construct a $n_d \times n_d$ matrix $\mathbf{M}_d$ with the following structure:

$$\mathbf{M}_9 = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_0 & \mathbf{B}_0 \\ \mathbf{C}_0 & \mathbf{D}_0 \end{bmatrix} - \lambda \begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad (8)$$

where $[\mathbf{A}\ \mathbf{B}]$ consists of a subset of the rows of $\mathbf{E}_d(\mathbf{c}, \lambda)$, and $[\mathbf{C}\ \mathbf{D}]$ consists of a subset of the rows of $\mathbf{F}_d(\mathbf{c})$. In particular, $\text{size}(\mathbf{A}) = 40 \times 40$, $\text{size}(\mathbf{B}) = 40 \times (n_d - 40)$, $\text{size}(\mathbf{C}) = (n_d - 40) \times 40$ and $\text{size}(\mathbf{D}) = (n_d - 40) \times (n_d - 40)$, and furthermore, $\text{rank}(\mathbf{M}_d) = n_d$, $\text{rank}(\mathbf{A}) = 40$, and $\text{rank}(\mathbf{D}) = n_d - 40$ for a general choice of λ.

The solution of the pose estimation problem is then to find λ such that

$$\mathbf{M}_d\ \mathbf{m}_d^T = 0 \quad (9)$$

which is possible if $\det(\mathbf{M}_d) = 0$. This leads to solving the generalised eigenvalue problem

$$\det(\mathbf{Q}_0 + \lambda \mathbf{Q}_1) = 0 \quad (10)$$

where

$$\mathbf{Q}_i = \mathbf{A}_i - \mathbf{B}_i \mathbf{D}_0^{-1} \mathbf{C}_0,\ \ i = 0, 1 \quad (11)$$

For any d such that the conditions 1) and 2) are satisfied, the problem amounts to solve a 40×40 generalised eigenvalue problem. The costly operation is the inversion of the block $\mathbf{D}_0$. In the next section, we explain the choice $d = 9$ of [10] and propose a method which allows us to solve the pose estimation problem using the hidden variable approach for $d < 9$. This reduces the size of the matrix $\mathbf{M}_d$ and hence of the block $\mathbf{D}_0$, leading to lower computational times.

III. Closed-form solutions with Sylvester forms

A. Saturation

Definition 4: The saturation of the graded ideal I with respect to the ideal $\mathfrak{m} := (w, x, y, z)$ is the ideal

$$I^{\text{sat}} := \{p \in \mathbb{C}[\lambda][w, x, y, z] \text{ s. t. } \exists n \in \mathbb{N} : \mathfrak{m}^n p \subseteq I\} \quad (12)$$

From the definition, we have $I \subseteq I^{\text{sat}}$ and $V(I) = V(I^{\text{sat}})$ (including their local algebraic structures, e.g. multiplicities), see [6, Lecture 5] for more details. In particular, $(I)_d = (I^{\text{sat}})_d$ for $d \gg 0$. Given an integer d, consider the matrix $\mathbf{E}_d(\mathbf{c}, \lambda)$. In elimination theory, it is well known that the two conditions 1) and 2) hold for any d such that $(I)_d = (I^{\text{sat}})_d$ (see e.g. [2, Theorem 3.20]). In addition, since I is generated by 4 equations of degree 3 in w, x, y, z, this properties hold if $d \geq 4(3-1)+1 = 9$ (see e.g. [3, Lemma 2.2]). This means that degree 9 is the minimum degree for which the matrix $\mathbf{E}_d(\mathbf{c}, \lambda)$ has the expected properties, explaining why $d = 9$ is used in [10].

To overcome the limitation $d \geq 9$, we must introduce new equations to build a new ideal that is saturated in a smaller degree. Our strategy is to take those equations in I^{sat} so that the solution set $V(I)$ is unchanged. In addition, since we are targeting closed-form solutions, those equations must be given in closed-form in the coefficients $\mathbf{c}$. For that purpose, we use Sylvester forms that have been initially introduced in [8] (see also [3, §2.10]).

B. Sylvester forms

Given $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \mathbb{N}^4$, we set $|\alpha| = \sum_{i=1}^4 \alpha_i$ and we define

$$\mathbf{q}^\alpha := [w^{\alpha_1+1}, x^{\alpha_2+1}, y^{\alpha_3+1}, z^{\alpha_4+1}] \quad (13)$$

Since the polynomials e_1, e_2, e_3 and e_4 are homogeneous of degree 3 with respect to $\mathbf{q}$, for any α such that $|\alpha| < 3$, it is possible to find decompositions

$$e_i = [h_{i,1}, h_{i,2}, h_{i,3}, h_{i,4}]\ (\mathbf{q}^\alpha)^T,\ \ i = 1, \ldots, 4 \quad (14)$$

where $h_{i,j}(\mathbf{q}, \lambda) \in \mathbb{C}[\lambda][w, x, y, z]$ are homogeneous polynomials of degree $3 - \alpha_j - 1$ in $\mathbf{q}$.

Definition 5: For any $\alpha \in \mathbb{N}^4$ such that $|\alpha| < 3$, the determinant

$$S_\alpha = \det\left((h_{i,j})_{i,j=1,\ldots,4}\right) \tag{15}$$

is called a Sylvester form (of $e_1, \ldots, e_4$ with respect to $\mathbf{q}$).

By construction, Sylvester forms depend on $\mathbf{q}, \lambda$ and $\mathbf{c}$ and they belong to I^{sat}. Furthermore, S_α is homogeneous of degree $8 - |\alpha|$ in $\mathbf{q}$, homogeneous of degree 4 in $\mathbf{c}$ and λ and is of degree at most 4 in λ. Although Sylvester forms depend on decompositions (14), they are unique modulo elements in the ideal I (see [3, Proposition 2.11]).

As a consequence, Sylvester forms can be added to the ideal I to obtain a new ideal that has the same saturation but being itself saturated in a smaller degree. We exploit this property to build new closed-form solutions to the problem of pose estimation.

C. Closed-form solution in $d < 9$

For $d = 8$, there exists one class of Sylvester forms modulo I of homogeneous degree 8 in $\mathbf{q}$. For $d = 7$, there are 4 classes of Sylvester forms modulo I, each of them of homogeneous degree 7 in $\mathbf{q}$. In both cases, the Sylvester forms can be chosen such that they are linear in λ and the coefficients at the monomials $m \in \mathbf{m}_d$, $d = 7, 8$, are determinants of 4×4 and 3×3 submatrices of the 4×20 coefficient matrix of the equations (1). Hence the Sylvester forms can be evaluated efficiently. The vectors of the coefficients of the Sylvester forms are added to the coefficient matrix $\mathbf{E}_d$, resulting in a matrix $\mathbf{E}'_d$, $d = 7, 8$, which has full rank for a general choice of λ and satisfies the conditions 1) and 2). The solutions of the system (1) can be obtained by solving the generalised eigenvalue problem, following the procedure described in (9)-(11). A more detailed description can be found in [12].

Reducing the degree leads to faster computational times. The most expensive operation of the solver is the inversion of the block $\mathbf{D}$ in (11), which has complexity $\mathcal{O}(n^3)$ for a $n \times n$ matrix. The size of the block $\mathbf{D}$ is $(n_d - 40) \times (n_d - 40)$, see the second column of Table I.

From the remaining monomials, we chose 24, which, together with the fixed 16 monomials, index the columns of the blocks $\mathbf{A}$ and $\mathbf{C}$. The remaining $n_d - 40$ monomials index the columns of $\mathbf{B}$ and $\mathbf{D}$. The third column of Table I shows the number of possible choices for $d = 9, 8, 7$. We currently do not have a principled method for selecting the ordering that gives the best numerical conditioning, but we found that *degrevlex* monomial ordering provide good results. Furthermore, the numerical accuracy depends on the ordering of columns of the matrix $\mathbf{M}_d$, which are indexed by the monomials $\mathbf{m}_d$. In order to avoid degeneracy when recovering the solutions, we fix the first 16 monomials to be

$$w^d, w^{d-1}x, w^{d-1}y, w^{d-1}z, x^d, wx^{d-1}, \ldots, yz^{d-1}, z^d$$

TABLE I: Lower degrees lead to more efficient solvers and considerably reduce the the number of possible choices for column ordering.

d	$n_d - 40$	$\binom{n_d-16}{24}$
9	180	1.0639×10^{31}
8	125	3.2608×10^{27}
7	80	2.3193×10^{23}

IV. Experiments

Similarly to previous works [10], [14], we used the KITTI dataset [5] for the experimental evaluation. The current set of 3D points (LiDAR scan $k{+}1$) is segmented to extract planar structures and matched with the closest 3D points from the reference set (LiDAR scan k). The pose estimated with state-of-the art approaches [10], [13], [14] and the methods proposed in this paper are compared with the ground truth. Table II shows the mean and standard deviation of translation and rotation errors and computation time obtained on sequences 03 and 07 of the KITTI dataset. More experiments can be found in [12]. The numerical accuracy of the proposed methods depends on the ordering of columns. For example, we observed that the ordering derived from *degrevlex* monomial ordering leads to more precise results that the one derived from *lex* ordering.

TABLE II: Comparison on KITTI dataset.

KITTI sequence 03 (800 frames)			
method	rotation (°) $\mu(\delta r) \pm \sigma(\delta r)$	translation (m) $\mu(\delta t) \pm \sigma(\delta t)$	time (ms) $\mu(t) \pm \sigma(t)$
Malis [10]	0.2741±6.3519	0.0237±0.2125	3.3211±0.9909
Wientapper [13]	0.0521±0.0365	0.0163±0.011	62.893±12.574
Zhou [14]	0.0746±0.6813	0.018±0.0481	2.9928±0.9365
deg8	**0.0493±0.0162**	**0.0181±0.0115**	3.0317±0.744
deg7	**0.0493±0.0162**	**0.0181±0.0115**	**2.5318±0.4278**

KITTI sequence 07 (1100 frames)			
method	rotation (°) $\mu(\delta r) \pm \sigma(\delta r)$	translation (m) $\mu(\delta t) \pm \sigma(\delta t)$	time (ms) $\mu(t) \pm \sigma(t)$
Malis [10]	1.1871±14.299	0.0866±0.9645	3.5282±0.9402
Wientapper [13]	0.0467±0.0391	0.0121±0.07	49.726±16.979
Zhou [14]	0971±1.0427	0.0136±0.0297	2.8219±0.6593
deg8	**0.0428± 0.0335**	**0.012±0.007**	3.498±1.0466
deg7	**0.0428± 0.0335**	**0.012±0.007**	**2.5303±0.7546**

V. Conclusion

We introduced Sylvester forms into robotics pose estimation and showed that they can be used to build efficient resultant-based closed-form solvers in degrees 7 and 8. Compared to the degree 9 hidden-variable construction, the proposed approach reduces elimination complexity, yields smaller matrices, and improves runtime while maintaining strong numerical performances. Beyond the specific algorithmic contribution, this work also suggests a hybrid perspective on geometric robotics. The solver itself is exact, interpretable, and derived from algebraic structure, yet its numerical robustness depends on implementation choices such as the column ordering that induces the block decomposition. We currently lack a principled theoretical criterion for choosing the ordering that provides the best numerical conditioning. Future work will therefore investigate hybrid geometric/data-driven strategies for selecting row subsets and learning column orderings from representative instances, with the goal of improving conditioning, robustness, and runtime without replacing the underlying geometric solver.

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